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Maximal couplings in $\mathcal{PT}$-symmetric chain models with the real spectrum of energies

Miloslav Znojil

Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic

E-mail: znojil@ujf.cas.cz

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Abstract

The domain $\mathcal{D}$ of all the coupling strengths compatible with the reality of the energies is studied for a family of non-Hermitian $N \times N$ matrix Hamiltonians $H^{(N)}$ with tridiagonal and $\mathcal{PT}$-symmetric structure. At all dimensions N, the coordinates are found of the extremal points at which the boundary hypersurface $\partial\mathcal{D}$ touches the circumscribed sphere (for odd $N = 2M + 1$) or ellipsoid (for even $N = 2K$).

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1. Introduction

1.1. Non-Hermitian chain models

In many quantum systems (typically, in nuclear and condensed matter physics), the observed spectra can be fitted by the equidistant harmonic-oscillator energies (i.e., by $E_n^{(\mathrm{HO})} = 2n+1$ in suitable units). An improvement of this fit can be based on a perturbatively mediated transition, say, to the popular nearest-neighbour-interaction model with an infinite-dimensional 'chain-model' tridiagonal Hamiltonian

$$H^{(\infty)} = \begin{bmatrix} 1 & a_0 & 0 & \dots & \\ b_0 & 3 & a_1 & 0 & \dots \\ 0 & b_1 & 5 & a_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1}$$

For the real coupling strengths a_n, b_m the manifest asymmetry of our Hamiltonian $H^{(N)}$ (with infinite as well as finite matrix dimension N) need not necessarily contradict the postulates of quantum mechanics. This may be illustrated on the so-called Swanson's model with $N = \infty$ [1] or, more easily, on the simplest truncated two-dimensional special case of equation (1),

$$H^{(2)} = \begin{pmatrix} 1 & a \\ b & 3 \end{pmatrix}.$$

Both its eigenvalues $E_\pm = 2 \pm \sqrt{1+ab}$ remain real (i.e., in principle, 'observable') whenever $ab \geqslant -1$. Inside this 'domain of physical acceptability', i.e., for

$$(a, b) \in \mathcal{D}^{(2)} \equiv \{(x, y)|x, y \in \mathbb{R}, xy > -1\} \tag{2}$$

these energies are also non-degenerate. This gives the technically welcome guarantee that $H^{(2)}$ can be diagonalized in a bi-orthogonal basis formed by the two respective sets of the right and left eigenvectors $|\pm\rangle$ and $|\pm\rangle\rangle$ such that

$$H^{(2)}|\pm\rangle = E_\pm|\pm\rangle, \qquad \langle\langle\pm|H^{(2)} = \langle\langle\pm|E_\pm. \tag{3}$$

The diagonalizability is lost on the boundary $\partial\mathcal{D}^{(2)}$ (where the basis of equation (3) becomes incomplete) and the reality of the energies is lost everywhere in the open complement of $\mathcal{D}^{(2)}$.

In a way discussed thoroughly in our recent letter [2], the two-dimensional model (3) proves particularly useful for an elementary explicit illustration of one of the 'key tricks' which re-assigns the necessary Hermiticity to the similar operators. The goal is being achieved by a suitable redefinition of the metric Θ and, hence, of the scalar product,

$$|\psi\rangle \odot |\phi\rangle \equiv \langle\psi|\Theta|\phi\rangle, \qquad \Theta = \Theta^\dagger > 0. \tag{4}$$

During the last few years, such a recipe has been revealed and/or implemented by several independent groups of authors sampled here in [3–6].

1.2. Construction of the metrics Θ for a given Hamiltonian

One should re-emphasize that, in general, a redefinition of the metric Θ in Hilbert space is fully compatible with the postulates of quantum mechanics, provided only that the reality of the spectrum is guaranteed. From time to time, the efficiency of the trick is being confirmed in various less standard applications of quantum theory [7, 8].

In the notation of equation (3), the essence of the trick derives from the observation that for many manifestly non-Hermitian Hamiltonians $H \neq H^\dagger$ with real spectra one can follow the two-dimensional guidance and construct the two families of the left eigenvectors $|n\rangle\rangle$ and of the right eigenvectors $|m\rangle$ of a given H. They form a bi-orthogonal basis in Hilbert space. In the next step one easily verifies, in all the finite-dimensional cases at least, that the operator defined by the spectral-representation-like formal expansion

$$\Theta = \sum_n |n\rangle\rangle s_n \langle\langle n| \tag{5}$$

satisfies the linear operator equation

$$H^\dagger\Theta = \Theta H. \tag{6}$$

In the final step of the argument one restricts *all* the parameters s_n to the real and positive numbers and concludes that the properties of the resulting operator Θ qualify it for a metric-operator interpretation as discussed in the review paper [3]. This means that the 'correct' inner product is ambiguous as it may be defined by *any* prescription (5). Its choice in fact fixes our selection of an explicit representation of the Hilbert space of states and, hence, 'the physics'.

There exist several remarkable differences between the unique, 'standard' choice of $\Theta = I$ and all the 'nonstandard' $\Theta \neq I$ in equation (4). For this reason, usually, the Hermiticity condition (6) with $\Theta \neq I$ is being re-named to 'quasi-Hermiticity' [3, 5]. One of the most characteristic consequences of the quasi-Hermiticity of a Hamiltonian H lies in the necessity of a specification of the domain $\mathcal{D}$ of parameters where the spectrum of energies remains real.

2. $\mathcal{PT}$-symmetric models

2.1. Modified harmonic oscillators

Under the assumption $H \neq H^\dagger$ some of the eigenvalues become complex whenever we leave the quasi-Hermiticity domain $\mathcal{D}$ of parameters in H. In the context of one-dimensional differential Schrödinger operators the problem has been made popular by Bender *et al* [4] who studied the generalized Bessis' oscillators

$$H^{(GB)}(\nu) = -\frac{d^2}{dx^2} + g(x)x^2, \qquad g(x) = (ix)^\nu, \qquad \nu \in \mathbb{R} \tag{7}$$

and conjectured that all the bound-state energies remain real iff $\nu \geqslant 0$, i.e., inside the very large domain $\mathcal{D}^{(GB)} \equiv (0, \infty)$ of the exponents ν. Rigorously, this conjecture has only been proved three years later [9]. One should note that the difficulty of this proof is in a sharp contrast with the elementary character of the above-mentioned construction of $\mathcal{D}^{(2)}$ related to the finite-dimensional $H^{(2)}$.

Our present paper is inspired by the question of feasibility of the constructions of the quasi-Hermiticity domains $\mathcal{D}^{(N)}$ for matrices at the higher dimensions $N > 2$. Predecessors of such a project can be seen not only in the exhaustive analyses of virtually all the two-dimensional cases [10] but also in our recent note [11] where we reported the feasibility of a complete and non-numerical reconstruction of the domain $\mathcal{D}^{(3)}$ for certain special $\mathcal{PT}$-symmetric three-by-three toy Hamiltonians.

We shall address here the natural question of the specification of $\mathcal{D}$ for the matrix family of the perturbed harmonic-oscillator Hamiltonians (1) restricted by an additional requirement of their $\mathcal{PT}$-symmetry. We believe that such band-matrix models are really exceptional. One of our reasons originates from the observation that in the most elementary differential-operator representation of $H^{(HO)}$, all the wavefunctions $\psi_n(x)$ pertaining to the above-listed energies $E_n^{(HO)} = 2n + 1$ are endowed with an additional, parity quantum number, $\mathcal{P}\psi_n(x) = \psi_n(-x) = (-1)^n\psi_n(x)$. This is a consequence of the commutativity $\mathcal{P}H^{(HO)} = H^{(HO)}\mathcal{P}$ which is manifestly broken in all the perturbed matrix models $H^{(N)}$.

The $\mathcal{PT}$-symmetry requirement $\mathcal{PT}H^{(N)} = H^{(N)}\mathcal{PT}$ is quite natural to impose, especially because the operator $\mathcal{T}$ can be treated as a mere transposition. In addition, it is easy to imagine that in the given basis the operator $\mathcal{P}$ is represented by the diagonal matrix with elements $\mathcal{P}_{nn} = (-1)^n$. As a net consequence, the requirement of the $\mathcal{PT}$-symmetry degenerates to the elementary rule $a_n = -b_n$ at all subscripts n in equation (1),

$$H^{(N)} = \begin{bmatrix} 1 & a_0 & 0 & \dots & 0 \\ -a_0 & 3 & a_1 & \ddots & \vdots \\ 0 & -a_1 & 5 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-2} \\ 0 & \dots & 0 & -a_{N-2} & 2N-1 \end{bmatrix}. \tag{8}$$

These are the models which we are going to analyse.

2.2. An additional 'up–down' symmetrization

2.2.1. A generic attraction of the levels. After a few numerical experiments with equation (8) one reveals a comparatively robust survival of the reality of the spectrum in perturbative regime. The phenomenon can be understood as one of the mathematically most interesting consequences of the 'sufficient separation' of the matrix elements $1, 3, \ldots$ along the main diagonal [12].

In contrast, even a strict observation of the equidistance of the elements on the main diagonal need not necessarily be of any help in a deeply nonperturbative regime. This danger is well known and the monograph [13] can be consulted for an extremely persuasive illustration of the emergence of unexpected difficulties even in a *symmetric* nonperturbative version of our example (8) with an apparently innocent choice of the dimension $N = 20$ and with an apparently 'not too nonperturbative' constant diagonal where $a_0 = a_1 = \cdots = a_{18} = a$.

In order to avoid similar complications in our present $\mathcal{PT}$-symmetric models we may try to assume, in the first step, that just a single coupling $a = a_k$ becomes large. In such a case, solely the two neighbouring energies become involved and perceivably modified. Generically, in a way controlled by the mere two-dimensional submatrix $H^{(2)}_{(k)}$ of $H^{(N)}$ we have

$$H^{(2)}_{(k)} = \begin{pmatrix} 2k+1 & a \\ -a & 2k+3 \end{pmatrix}$$

so that the energy values become 'attracted' by each other in proportion to $|a_k|$ at any $k < N-1$,

$$E_k = 2k + 2 - \sqrt{1-a^2}, \qquad E_{k+1} = 2k + 2 + \sqrt{1-a^2}.$$

The mechanism of this effect is virtually independent of the rest of the spectrum (which may be considered pre-diagonalized) so that we may *always* expect that some energies get complexified *whenever* the couplings become sufficiently strong.

This means that, intuitively, we may always visualize the coupling dependence of the energies as their *mutual* attraction. In this sense we are able to guess that the levels E_{n_0} in the middle of the matrix (i.e., such that $n \approx n_0 \approx N/2$) will be 'maximally protected' against the complexification due to their multiple and balanced 'up' and 'down' attraction by all the other levels.

Such a balance may be quantitatively (though not qualitatively) violated by the differences in the absolute values of the pairs of couplings a_{n_0+k} and a_{n_0-k} at all the allowed index shifts k. For this reason, we shall restrict our present attention to the special class of the Hamiltonian matrices (8) which are, in this sense, symmetrized and have

$$a_j = a_{N-2-j} \qquad j = 0, 1, \ldots, j_{\max}(=\text{entier}[N/2]). \tag{9}$$

This means that everywhere in what follows we shall reduce the class of the $(N-1)$-parametric chain models (8) to its 'up–down-symmetrized' subset

$$H^{(N)} = \begin{bmatrix} \delta-1 & a_0 & 0 & \ldots & 0 \\ -a_0 & \delta-3 & \ddots & \ddots & \vdots \\ 0 & -a_1 & \delta-5 & a_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & a_0 \\ 0 & \ldots & 0 & -a_0 & \delta-2N+1 \end{bmatrix}. \tag{10}$$

Obviously, the specific choice of the global shift $\delta = N$ of the origin of the energy scale also makes the main diagonal of the whole matrix 'up–down' symmetric. Still, it is slightly unpleasant that at the strictly integer half-dimensions $K = N/2$ the 'last' free parameter $a_{j_{\max}}$ enters our matrix $H^{(N)}$ 'anomalously', i.e., just twice. This means that the parity of N introduces a fairly nontrivial difference between the corresponding up–down-symmetric models (10).

2.2.2. Even dimensions $N = 2K$, $K = 1, 2, \ldots$. As long as we intend to analyse the secular determinants of our matrices [13], it makes sense to simplify our notation and, in particular, to get rid of the subscripts and abbreviate $a = a_{j_{\max}}$, $b = a_{j_{\max}-1}$ and so on, up to the last element a_0 abbreviated, whenever needed, by the last letter z. In this notation, obviously, the symbol $z = a_0$ coincides with $a = a_{j_{\max}} = a_0$ at $K = 1$ in

$$H^{(2)} = \begin{bmatrix} 1 & a \\ -a & -1 \end{bmatrix}$$

or with b at $K = 2$ in

$$H^{(4)} = \begin{bmatrix} 3 & b & 0 & 0 \\ -b & 1 & a & 0 \\ 0 & -a & -1 & b \\ 0 & 0 & -b & -3 \end{bmatrix},$$

etc. We see that the general matrix (10) with $N = 2K$ may be easily understood as partitioned into four K-dimensional submatrices,

$$H^{(2K)} = \left[\begin{array}{cccc|cccc} 2K-1 & z & 0 & \ldots & & & & \\ -z & \ddots & \ddots & \ddots & \vdots & & & \\ 0 & \ddots & 3 & b & 0 & \ldots & & \\ \vdots & \ddots & -b & 1 & a & 0 & \ldots & \\ \hline & \ldots & 0 & -a & -1 & b & 0 & \ldots \\ & & \ldots & 0 & -b & -3 & \ddots & \\ & & & \vdots & \ddots & \ddots & \ddots & z \\ & & & & \ldots & 0 & -z & 1-2K \end{array}\right].$$

The simplest illustrative example $H^{(2)}$ has already been shortly discussed above (cf also [2]). In the general case the secular polynomial $\det(H^{(2K)} - E)$ will be a polynomial of the Kth degree in $s = E^2$ and it will only depend on the squares of the couplings $a^2_{j_{\max}} \equiv a^2 = A, a^2_{j_{\max}-1} \equiv b^2 = B, \ldots, a_0^2 \equiv z^2 = Z$.

2.2.3. Odd dimensions $N = 2M + 1$, $M = 1, 2, \ldots$. Whenever the dimension of our band-matrix Hamiltonian $H^{(N)}$ with equidistant matrix elements on its main diagonal is odd, $N = 2M + 1$, we have

$$H^{(2M+1)} = \left[\begin{array}{ccc|c|ccc} 2M & z & 0 & 0 & 0 & 0 & 0 \\ -z & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & \ddots & 2 & a & 0 & 0 & 0 \\ \hline 0 & 0 & -a & 0 & a & 0 & 0 \\ \hline 0 & 0 & 0 & -a & -2 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & z \\ 0 & 0 & 0 & 0 & 0 & -z & -2M \end{array}\right].$$

Here the central matrix element vanishes and the $\mathcal{PT}$-symmetric coupling is mediated again by the M real matrix elements $a, b, \ldots, z$. Omitting the overall factor E we may reduce the secular polynomial $\det(H^{(2M+1)} - E)$ to a polynomial of the Mth degree in $s = E^2$. It will again depend on the squares of the couplings only.

3. Hamiltonians of the even dimensions $N = 2K$

In the two-dimensional case with $K = 1$ the whole discussion remains entirely elementary (see above) and one can conclude that there exist precisely two points of the boundary $\partial\mathcal{D}^{(2)}$ (called 'exceptional points' in the literature [14, 15]) which are defined by the elementary rule $a_{\pm}^{(\mathrm{EP})} = \pm 1$, i.e., by the single root $A^{(\mathrm{EP})} = 1$ of the single energy-degeneracy condition.

3.1. Four-by-four model, $K = 2$

For the four-by-four Hamiltonian $H^{(4)}$ the standard definition of the spectrum

$$\det \begin{bmatrix} 3-E & b & 0 & 0 \\ -b & 1-E & a & 0 \\ 0 & -a & -1-E & b \\ 0 & 0 & -b & -3-E \end{bmatrix} = 0,$$

i.e., the quadratic secular equation for $s = E^2$,

$$s^2 + (-10 + 2b^2 + a^2)s + 9 + 6b^2 - 9a^2 + b^4 = 0,$$

can easily be solved in closed form,

$$s = s_{\pm} = 5 - b^2 - 1/2a^2 \pm 1/2\sqrt{64 - 64b^2 + 16a^2 + 4b^2a^2 + a^4}. \tag{11}$$

These formulae may be read as an implicit definition of $\mathcal{D}^{(4)}$, i.e., of the reality domain of the energies or, equivalently [5], of the quasi-Hermiticity domain of the Hamiltonian of our $K = 2$ chain model.

For a more explicit construction of $\mathcal{D}^{(4)}$ we can make use of the up–down symmetry (9) and imagine that during the initial perturbative mutual attraction of the neighbouring levels one can only guarantee the growth of the ground-state minimum $E_0 = E_{-,+} \equiv -\sqrt{s_+}$ and the decrease of the top-state maximum $E_3 = E_{+,+} \equiv +\sqrt{s_+}$.

Beyond perturbative domain, at certain 'exceptional-point' combinations $(a, b) = (a, b)^{(\mathrm{EP}_+)}$ of the sufficiently large strengths a and b, the latter two extreme energy levels will ultimately coincide (and, immediately afterwards, complexify) in a way discussed in section 2.2.1, $E_{-,+}^{(\mathrm{EP})} = E_{+,+}^{(\mathrm{EP})} = 0$. At another set of the EP coupling doublets $(a, b) = (a, b)^{(\mathrm{EP}_-)}$, both the two 'internal' levels may also coincide as well, $E_{\pm,-} \equiv \pm\sqrt{s_-} = 0$. In this way, the complete boundary $\partial\mathcal{D}^{(4)}$ of the quasi-Hermiticity domain is a curve in the a–b plane formed by the 'weaker' doublets of the EP strengths $(a, b)^{(\mathrm{EP}_\pm)}$. The shape of such a boundary can be deduced from equation (11) (cf figure 1).

In a magnified detail, figure 2 demonstrates that the graphical representation ceases to be reliable in the fairly large vicinity of the common maximum of the sizes of the allowed couplings a and b. Fortunately, near any such an 'extremely exceptional' point $(a, b) = \{(\pm\sqrt{A^{(\mathrm{EEP})}}, \pm\sqrt{B^{(\mathrm{EEP})}}\}$ of the a–b plane, the details of the shape of the boundary $\partial\mathcal{D}^{(4)}$ can be described by the purely analytic means. An extension of the latter observation to all the dimensions N will become, after all, a core of our present message.

Let us explain the method for $N = 2K$ at any K. In the first step one realizes that $s^{(\mathrm{EEP})} = 0$ due to the up–down symmetry. As long as this must be the only root (i.e., a maximally degenerate root) of the polynomial secular equation

$$s^K + P_{K-1}(A, B, \ldots)s^{K-1} + P_{K-2}(A, B, \ldots)s^{K-2} + \cdots = 0, \tag{12}$$

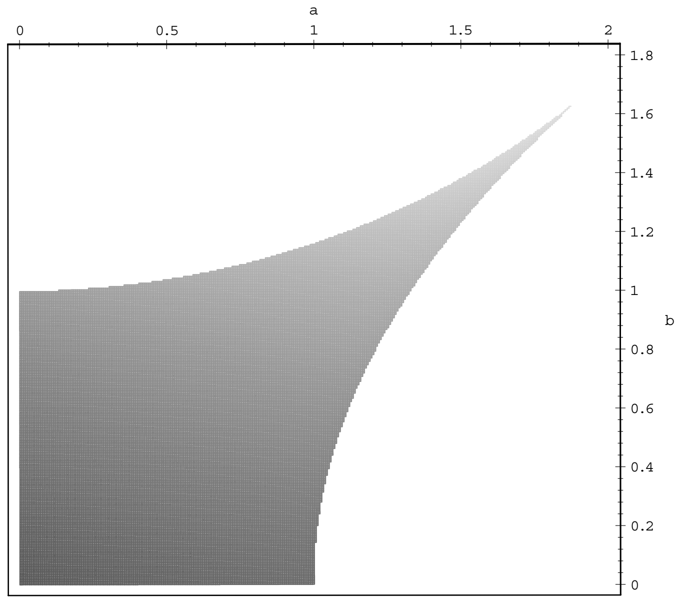

Figure 1. One quarter of the domain $\mathcal{D}^{(4)}$ (cf section 3.1).

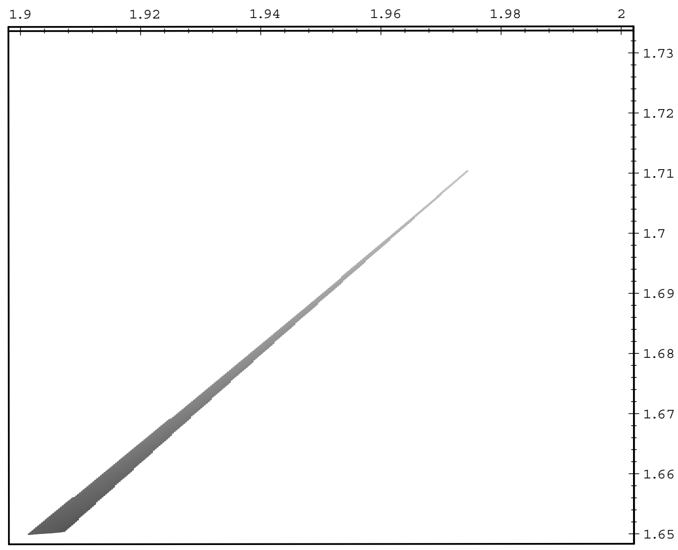

Figure 2. A magnified spike of the domain $\mathcal{D}^{(4)}$.

it can only exist if the K values $A^{(\mathrm{EEP})}, B^{(\mathrm{EEP})}, \ldots$ of the EEP coupling strengths satisfy the nonlinear set of the following K necessary conditions:

$$\begin{aligned}
&P_{K-1}(A^{(\mathrm{EEP})}, B^{(\mathrm{EEP})}, \ldots) = 0, \\
&P_{K-2}(A^{(\mathrm{EEP})}, B^{(\mathrm{DEEP})}, \ldots) = 0, \\
&\ldots \\
&P_0(A^{(\mathrm{EEP})}, B^{(\mathrm{EEP})}, \ldots) = 0.
\end{aligned} \tag{13}$$

At $K = 2$ the latter set of polynomial equations reads

$$A + 2B = 10, \qquad (3 + B)^2 = 9A$$

and an elimination of A leads to a quadratic equation for $B + 3$ giving a spurious solution $A = 64$ and $B = -27$ (which would imply an imaginary coupling b) and the unique correct solution $A^{(\mathrm{EEP})} = 4$ and $B^{(\mathrm{EEP})} = 3$.

3.2. Six by six model, $K = 3$

In a full parallel with the preceding subsection, secular equation

$$\det \begin{bmatrix} 5-E & c & 0 & 0 & 0 & 0 \\ -c & 3-E & b & 0 & 0 & 0 \\ 0 & -b & 1-E & a & 0 & 0 \\ 0 & 0 & -a & -1-E & b & 0 \\ 0 & 0 & 0 & -b & -3-E & c \\ 0 & 0 & 0 & 0 & -c & -5-E \end{bmatrix} = 0$$

in its polynomial form (12),

$$s^3 + (2b^2 - 35 + 2c^2 + a^2)s^2 + (b^4 + 2c^2a^2 - 44b^2 + 28c^2 - 34a^2 + c^4 + 259 + 2b^2c^2)s \\ + a^2c^4 - 10b^2c^2 + 30c^2a^2 + 225a^2 - 30c^2 - c^4 - 25b^4 - 225 - 150b^2 = 0,$$

remains solvable in closed form. As long as our present attention is concentrated on the EEP extremes, we shall skip the details of the complete description of the hedgehog-shaped surface $\partial\mathcal{D}^{(6)}$ in the full three-parametric space and note only that this shape must be all contained within the ellipsoid with the boundary described by the first constraint of equation (13), $a^2 + 2b^2 + 2c^2 = 35$.

At $K = 3$ the full solution of the triplet of equation (13) ceases to be easy but it still remains feasible. Besides the unique and acceptable correct solution

$$A^{(\mathrm{EEP})} = 9, \qquad B^{(\mathrm{EEP})} = 8, \qquad C^{(\mathrm{EEP})} = 5, \qquad K = 3, \tag{14}$$

one obtains another set of the alternative solutions generated, after the patient elimination of A and B, in terms of roots of a final 'effective' polynomial in single variable C,

$$416C^4 + 20\,909C^3 + 22\,505C^2 + 28\,734\,375C - 48\,828\,125 = 0. \tag{15}$$

Out of its two real roots, $C_- = -65.803\,607\,06$ and $C_+ = 1.693\,394\,621$, the former one is manifestly spurious giving the imaginary coupling c. For the latter root we have to recall the corresponding condition

$$22\,156\,250B_+ + 2912{C_+}^3 + 1\,446\,363{C_+}^2 + 820\,546\,875 + 9\,654\,410C_+ = 0$$

to see that the coupling $b = \sqrt{B_+}$ is imaginary and should be rejected as spurious as well.

3.3. Eight by eight model, $K = 4$

Out of the four EEP constraints (13) at $K = 4$ the first equation $P_3(a^2, b^2, c^2, d^2) = 0$ defines the surface of an ellipsoid or, after the change of variables $a \to A = a^2$ etc, a planar side of a simplex,

$$A + 2B + 2C + 2D = 84.$$

By construction, the domain $\mathcal{D}^{(8)}$ is circumscribed by this ellipsoid or simplex. Unfortunately, one hardly finds any immediate geometric interpretation of the remaining quadratic, cubic and quartic polynomial equations $P_2(A, B, C, D) = 0$, $P_1(A, B, C, D) = 0$ and $P_0(A, B, C, D) = 0$ containing 13, 19 and 20 individual terms, respectively, and admitting just marginal simplifications, e.g., to the nine-term equation

$$1974 + (B + C + D)^2 + 2AD + 2BD + 2AC = 83A + 142B + 70C - 50D$$

in the P_2-case, etc.

In this setting it comes as a real surprise that the above-derived $K = 3$ rule (14) still finds its unique $K = 4$ counterpart which, in addition, possesses the closed form again,

$$A^{(\mathrm{EEP})} = 16, \qquad B^{(\mathrm{EEP})} = 15, \qquad C^{(\mathrm{EEP})} = 12, \; D^{(\mathrm{EEP})} = 7, \qquad K = 4. \tag{16}$$

Its derivation necessitated the use of the fully computer-assisted Gröbner-basis technique. Just for illustration one may mention the $K = 4$ form of the final 'effective' polynomial equation,

$$314\,432D^{17} - 5\,932\,158\,016D^{16} + \cdots$$
$$+ 153\,712\,881\,941\,946\,532\,798\,614\,648\,361\,265\,167 = 0,$$

representing the 'next-door neighbour' of the still exactly factorizable equation (15).

In a test of the uniqueness of solution (16) one finds out that it possesses seven real and positive roots D. Out of them, the following three ones are negative and, hence, manifestly spurious, $-203.914\,709\,5$, $-156.666\,700\,1$, $-55.499\,924\,41$. We skipped the proof of the spuriosity for the remaining four roots, namely, of $0.419\,285\,438\,5$, $5.354\,156\,128$, $1354.675\,195$ and $18\,028.167\,89$ since the related calculations, however straightforward, become unpleasant and clumsy. For example, the values of A are given by the rule $\alpha \times A =$ (a polynomial in D of 16th degree) where the number of digits in the auxiliary integer constant α exceeds one hundred.

3.4. Arbitrary even dimension $N = 2K$

Even though we did not dare to test the applicability of the Gröbner-basis technique at $K = 5$, we were lucky in noticing that the previous results already admitted the following extrapolation *to any* K:

$$A^{(\mathrm{EEP})} = K^2, \qquad B^{(\mathrm{EEP})} = K^2 - 1^2, \qquad C^{(\mathrm{EEP})} = K^2 - 2^2, \qquad D^{(\mathrm{EEP})} = K^2 - 3^2, \ldots. \tag{17}$$

This is our first main result. The validity of this empirically revealed rule has subsequently been tested and confirmed by the incomparably simpler direct insertions.

As a byproduct of these verifications, the general ellipsoidal surface form of the first item in equation (13) has been predicted from the data available at $K \leqslant 4$ and re-confirmed at several higher $K > 4$ giving, in terms of the original coupling-strength variables of equation (8) with symmetry (9),

$$A + 2\,(B + C + \cdots + Z) \equiv a_{j_{\max}}^2 + 2a_{j_{\max}-1}^2 + \cdots + 2a_0^2 = \sum_{k=0}^{N-2} a_k^2 = \frac{4K^3 - K}{3} \tag{18}$$

or, in the form of an immersion of $\mathcal{D}$ in an ellipsoid in K dimensions,

$$a^2 + 2b^2 + \cdots + 2z^2 \equiv \sum_{k=0}^{N-2} a_k^2 \leqslant \frac{4K^3 - K}{3}. \tag{19}$$

These observations are in a complete agreement with the individually evaluated formulae and carry a geometric interpretation showing that every domain $\mathcal{D}^{(2K)}$ (where all the energies remain real) is circumscribed by a certain ellipsoidal hypersurface. Its intersections with the boundary $\partial\mathcal{D}^{(2K)}$ coincide with the 2^K EEP points with the coordinates $a^{(\mathrm{EEP})} = \pm K$, $b^{(\mathrm{EEP})} = \pm\sqrt{K^2 - 1}$, etc.

4. Hamiltonians of the odd dimensions $N = 2M + 1$

In a one-parametric three-by-three illustration with $M = 1$,

$$H^{(3)} = \begin{bmatrix} 2 & a & 0 \\ -a & 0 & a \\ 0 & -a & -2 \end{bmatrix}$$

the determination of the interval of quasi-Hermiticity $a \in \mathcal{D}^{(3)} = (-\sqrt{2}, \sqrt{2})$ is trivial since the secular equation $-E^3 + (4 - 2a^2)E = 0$ is exactly solvable. In a remark [11] we also studied a 'generic' three-dimensional (and three-parametric) matrix model where we relaxed both the equidistance assumption concerning the main diagonal *and* our present simplifying 'up–down' symmetrization assumption $a_0 = a_1$.

4.1. Five-by-five model, $M = 2$

A comparatively elementary two-parametric example of our present class of models of section 2.2.3 is still encountered at $M = 2$,

$$H^{(5)} = \begin{bmatrix} 4 & b & 0 & 0 & 0 \\ -b & 2 & a & 0 & 0 \\ 0 & -a & 0 & a & 0 \\ 0 & 0 & -a & -2 & b \\ 0 & 0 & 0 & -b & -4 \end{bmatrix}.$$

Its secular equation gives the central constant energy $E_0 = 0$. The other two pairs of the real or complex conjugate levels $E_n = -E_{-n} = \sqrt{s}$ with $n = 1, 2$ are obtained from the remaining polynomial equation in the new variable $s = E^2$,

$$-s^2 + (20 - 2b^2 - 2a^2)s - 64 - 16b^2 + 32a^2 - b^4 - 2a^2b^2 = 0. \tag{20}$$

We should determine the domain $\mathcal{D}^{(5)}$ in which all the energies remain real. This means that inside the closure of the domain of quasi-Hermiticity $\mathcal{D}^{(5)}$ both the roots of equation (20) must be non-negative.

Our task is elementary since the $M = 2$ eigenvalue problem is solvable in closed and compact form,

$$E_{\pm 1} = \pm\sqrt{10 - a^2 - b^2 - \sqrt{36 + 12a^2 + a^4 - 36b^2}},$$

$$E_{\pm 2} = \pm\sqrt{10 - a^2 - b^2 + \sqrt{36 + 12a^2 + a^4 - 36b^2}}.$$

Thus, the results of the method of preceding section may be complemented by direct calculations. In terms of the two non-negative quantities $A = a^2 \geqslant 0$ and $B = b^2 \geqslant 0$ the reality of the energies will be guaranteed by the triplet of inequalities. The first one reads $10 \geqslant A + B$ and restricts the allowed values of A and B to a simplex. The second condition $36 + 12A + A^2 \geqslant 36B$ requires that the allowed values of B must lie below a growing branch of a parabola $B_{\max} = B_{\max}(A)$. The third condition $(8 + B)^2 \geqslant (32 - 2B)A$ represents an easily visualized upper bound for $A \leqslant A_{\max} = A_{\max}(B)$ where the latter hyperbola-shaped function grows with B in all the interval of interest.

Beyond the above direct proof we may also parallel the considerations of the preceding section and imagine that the symmetry of equation (20) implies that its triple root must vanish, $s = s^{(\mathrm{EEP})} = 0$. This means that in the polynomial equation (20) both the coefficients at the subdominant powers of s must vanish. These two coupled conditions degenerate to the single quadratic equation with the unique non-spurious solution $A^{(\mathrm{EEP})} = 6$ and $B^{(\mathrm{EEP})} = 4$. Thus, in a way complementing our above direct discussion of the reality of the energies we see that at our EEP point all the three above-mentioned inequalities become saturated simultaneously.

4.2. Seven-by-seven model, M = 3

By the same Gröbner-basis method as above we derive the result

$$A^{(\mathrm{EEP})} = 12, \qquad B^{(\mathrm{EEP})} = 10, \qquad C^{(\mathrm{EEP})} = 6, \qquad M = 3. \tag{21}$$

It is again unique because one of the two roots $C_\pm = 27 \pm 9\sqrt{21}$ of the 'first alternative' Gröbnerian 'effective' equation $C^2 - 54C = 972$ and *both* the roots $-354 \pm 60\sqrt{34}$ of the 'second alternative' equation $C^2 + 708C + 2916 = 0$ are negative while the only remaining positive root $C_+ = 68.243\,181\,25$ gives the negative $B = 28 - 3C$.

4.3. All the (2M + 1)-dimensional models with M ⩾ 4

At $M = 4$ we still were able to evaluate the explicit form of the secular equation,

$$14\,745\,600 - 7\,372\,800A + \cdots + (-2C + 220 - 2B - 2A - 2D)\, s^4 - s^5 = 0$$

and we also still *computed* the $M = 4$ EEP solution directly,

$$A^{(\mathrm{EEP})} = 20, \qquad B^{(\mathrm{EEP})} = 18, \qquad C^{(\mathrm{EEP})} = 14, \qquad D^{(\mathrm{EEP})} = 8, \qquad M = 4. \tag{22}$$

We already gave up the discussion of its uniqueness as overcomplicated. Starting from $M = 5$ this enabled us to change the strategy and to continue the calculations by merely confirming the validity of the following general odd-dimensional formula:

$$\begin{aligned} A^{(\mathrm{EEP})} &= M(M+1), & B^{(\mathrm{EEP})} &= M(M+1) - 1\cdot 2 = M(M+1) - 2,\\ C^{(\mathrm{EEP})} &= M(M+1) - 2\cdot 3, & D^{(\mathrm{EEP})} &= M(M+1) - 3\cdot 4, \ldots. \end{aligned} \tag{23}$$

This formula is our second main result.

In order to complete the parallels with the previous section, let us finally recollect the universal ellipsoidal-surface embedding (18) of the domains $\mathcal{D}^{(2K)}$ and emphasize that its present odd-dimension analogue is even simpler. Indeed, returning once more to all the $M \leqslant 4$ calculations of this section we arrive at the extrapolation formula

$$A + B + C + D + \cdots + Z = \frac{2M^3 + 3M^2 + M}{3}, \tag{24}$$

the validity of which is very easily confirmed (and was confirmed) at a number of higher integers $M > 5$. Its alternative arrangement reads

$$a^2 + b^2 + \cdots + z^2 \leqslant \frac{2M^3 + 3M^2 + M}{3} \tag{25}$$

showing that every quasi-Hermiticity domain $\mathcal{D}^{(2M+1)}$ is circumscribed by a certain minimal hypersphere, with the mutual intersections lying precisely at the 2^M EEP points with the coordinates $a^{(\mathrm{EEP})} = \pm\sqrt{M(M+1)}$, $b^{(\mathrm{EEP})} = \pm\sqrt{M(M+1) - 2}$, etc.

5. Summary

We introduced a class of the tridiagonal and up–down symmetrized matrix chain models $H^{(N)}$, the spectrum of which remains equidistant in the decoupled limit. We believe that beyond their above-mentioned direct connection to physics of harmonic oscillators exposed to a small finite-dimensional perturbation, another interesting source of their possible future physical applicability could be sought in the equidistance of spectra of certain *manifestly finite-dimensional* spin-chain models possessing equidistant spectra (cf, for illustration, the Polychronakos' $SU(N)$ model [16] or its supersymmetric $SU(m|n)$ generalization [17] etc).

At any dimension N of our Hamiltonians $H^{(N)}$ we determined the coordinates of all the EEP (=extreme exceptional point) N-plets of the matrix elements $a^{(\mathrm{EEP})}$, $b^{(\mathrm{EEP})}$, ..., $z^{(\mathrm{EEP})}$, the choice of which leads to the maximal, N-fold degeneracy of the N-plet of the real energy levels pertaining to the underlying model. At $N = 2M + 1$ the latter EEP values are 'maximal' in the sense of the norm defined as a square root of the sum of their squares. The same comment applies at the even dimensions $N = 2K$ after a slight modification of the norm taking just one-half of the value of the 'central' coupling a^2 in the sum displayed in equation (18).

Some of the specific merits of our class of models may be seen:

- in the 'user-friendly' tridiagonal structure of its Hamiltonians $H^{(N)}$;
- in the feasibility of an illustrative simulation of *all* the possible scenarios leading to $2k$-tuple EP-like degeneracies of the energies (followed by their $\mathcal{PT}$-symmetry-related complexifications) at all the eligible multiplicities $k \leqslant N/2$;
- in the fact that for the latter and similar purposes the models contain *precisely* a necessary *and* sufficient number of free parameters;
- last but not least, in an 'exact solvability' leading to closed formulae at all the dimensions N, for the EEP coordinates at least.

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References

[1] Swanson M S 2004 *J. Math. Phys.* **45** 585
Geyer H B, Scholtz F G and Snyman I 2004 *Czech. J. Phys.* **54** 1069
[2] Znojil M and Geyer H B 2006 *Phys. Lett.* B **640** 52
[3] Scholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys., NY* **213** 74
[4] Bender C M and Milton K A 1997 *Phys. Rev.* D **55** R3255
Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 0270401
[5] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205, 2814
Wang Q 2004 *Czech. J. Phys.* **54** 143
Mostafazadeh A and Batal A 2004 *J. Phys. A: Math. Gen.* **37** 11645
Scolarici G and Solombrino L 2005 *Czech. J. Phys.* **55** 1177
Weigert S 2006 *J. Phys. A: Math. Gen.* **39** 10239
[6] Znojil M 2004 *Rendiconti del Circ. Mat. di Palermo, Ser. II, Suppl.* **72** 211 (*Preprint* math-ph/0104012)
Znojil M 2004 *Symmetry Methods in Physics CD ROM Proceedings (JINR, Dubna)* (*Preprint* hep-th/0408081)
Bagchi B, Quesne C and Znojil M 2001 *Mod. Phys. Lett.* A **16** 2047
Bíla H 2004 *Czech. J. Phys.* **54** 1049
Langer H and Tretter Ch 2004 *Czech. J. Phys.* **54** 1113
Hernández E, Járegui A and Mondragón A 2006 *J. Phys. A: Math. Gen.* **39** 10087
Krejčiřík D, Bíla H and Znojil M 2006 *J. Phys. A: Math. Gen.* **39** 10143
Znojil M 2006 *J. Phys. A: Math. Gen.* **39** 10247
[7] Buslaev V and Grecchi V 1993 *J. Phys. A: Math. Gen.* **26** 5541
Fernández F M, Guardiola R, Ros J and Znojil M 1998 *J. Phys. A: Math. Gen.* **31** 10105
Delabaere E and Pham F 1998 *Phys. Lett.* A **250** 25

[8] Bender C M 2005 *Czech. J. Phys.* **55** 1067
Fei S-H 2005 *Czech. J. Phys.* **55** 1085
Fei S-H 2006 *Czech. J. Phys.* **56** 887
Geyer H B and Snyman I 2005 *Czech. J. Phys.* **55** 1091
Günther U and Stefani F 2005 *Czech. J. Phys.* **55** 1099
Jones H F and Mateo J 2005 *Czech. J. Phys.* **55** 1117
Kleefeld F 2005 *Czech. J. Phys.* **55** 1123
Samsonov B F, Shamshutdinova V V and Gitman D M 2005 *Czech. J. Phys.* **55** 1123
Bagchi B, Banerjee A and Quesne C 2006 *Czech. J. Phys.* **56** 893
Figueira de Morisson Faria C and Fring A 2006 *Czech. J. Phys.* **56** 899
Mustafa O and Mazharimousavi S H 2006 *Czech. J. Phys.* **56** 967
Jakubský V and Smejkal J 2006 *Czech. J. Phys.* **556** 985
[9] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** 5679
Shin K C 2002 *Commun. Math. Phys.* **229** 543
[10] Mostafazadeh A and Ozcelik S 2006 *Turk. J. Phys.* **30** 437
[11] Znojil M 2007 *Phys. Lett.* B **647** 225
[12] Caliceti E, Cannata F and Graffi S 2006 *J. Phys. A: Math. Gen.* **39** 10019
[13] Wilkinson J H 1965 *The Algebraic Eigenvalue Problem* (Oxford: Oxford University Press) chapter 7
[14] Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer) p 64
[15] Heiss W D and Harney H L 2001 *Eur. Phys. J.* D **17** 149
[16] Polychronakos A P 1994 *Nucl. Phys.* B **419** 553
[17] Basu-Mallick B, Ujino H and Wadati M 1999 *J. Phys. Soc. Japan* **68** 3219